Internal Waves, Numerical Approximation of the Poincaré equation

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26th May 2003

Overview

Numerical approximation of waves in enclosed fluid volumes.

- **Physics**
- **Method of characteristics**
- **Discretisation**
- **Regularisation**
- **Results**

Internal waves

Internal gravity waves propagate through a fluid volume. Possible mechanisms **rotation** or **stratification**.
Applications include

- Flow in oceans or lakes, forced by tides.
- Leads to mixing and transport.
- Wave motion in the liquid core of the earth.

Numerical approximation is complicated by

- Ill-Posedness
- Small scale (fractal) features
- Large solution space
- The a-typical nature of the problem

Physics of stratified fluids

The linearised inviscous quasi-incompressible Boussinesq equations:

$$
\begin{aligned}
\rho^* p_t &= -p_x, \\
\rho^* p_t &= -p_y, \\
\rho^* w_t &= -p_z - \rho g, \\
\nabla \cdot \mathbf{u} &= 0, \\
\rho_t - w\frac{\rho^*}{g}N^2 &= 0
\end{aligned}
$$

At the boundary

$$\mathbf{u} \cdot \mathbf{n} = 0.$$

The Poincaré equation

- Time periodicity $[p, \mathbf{u}, \rho] \propto e^{i\omega t}$
- Bounded piecewise linear closed domain $\Omega \in R^2$.

After some calculation:

$$\begin{aligned} p_{xx} - \lambda^2 p_{zz} &= 0 \text{ in } \Omega, \\ (p_x, \sqrt{\lambda} p_z) \cdot \mathbf{n} &= 0 \text{ at } \partial\Omega \end{aligned}$$

or

$$\begin{aligned} \Psi_{xx} - \lambda^2 \Psi_{zz} &= 0 \text{ in } \Omega, \\ \Psi &= 0 \text{ at } \partial\Omega \end{aligned}$$

by introducing the streamfunction $\Psi_x = v$, $\Psi_z = -u$. We take λ as a *given* quantity from physics.

Properties of the Poincaré equation

Hyperbolic linear second order partial differential equation.
The *wave equation* in spatial coordinates only.

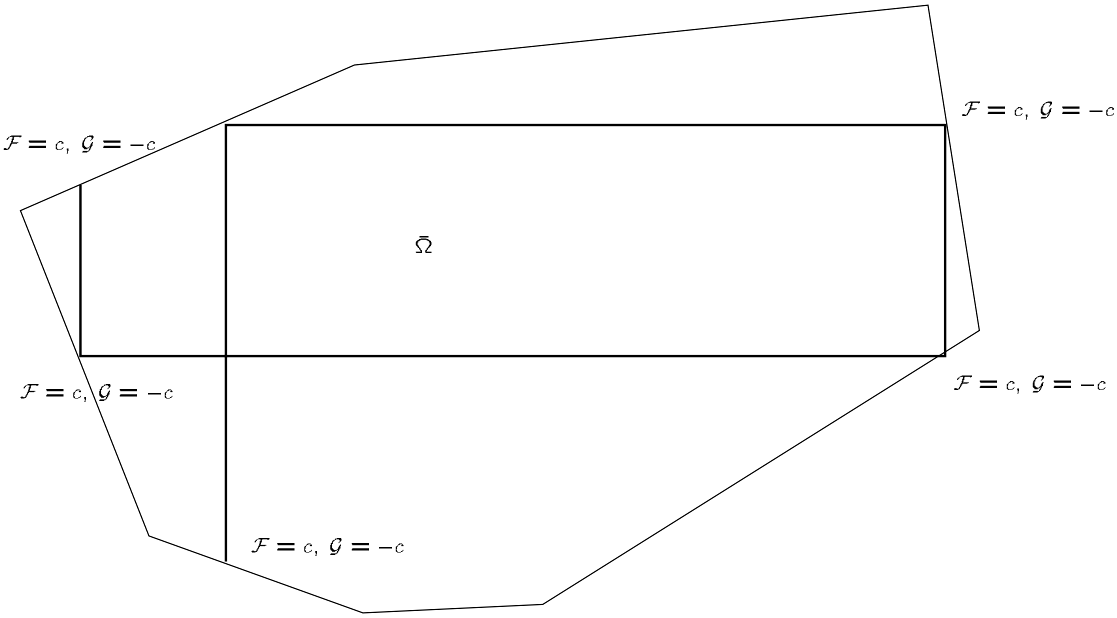
Consider the characteristic coordinates:

$$\begin{aligned}\xi &= x + \lambda z, \\ \eta &= x - \lambda z\end{aligned}$$

This gives $\Psi_{\xi\eta} = 0$ and leads to

$$\Psi(\xi,\eta) = \mathcal{F}(\xi) + \mathcal{G}(\eta) \quad \text{in} \quad \bar{\Omega},$$
$$\Psi(\xi,\eta) = 0 \quad \text{at} \quad \partial\bar{\Omega}.$$

Ill-Posedness

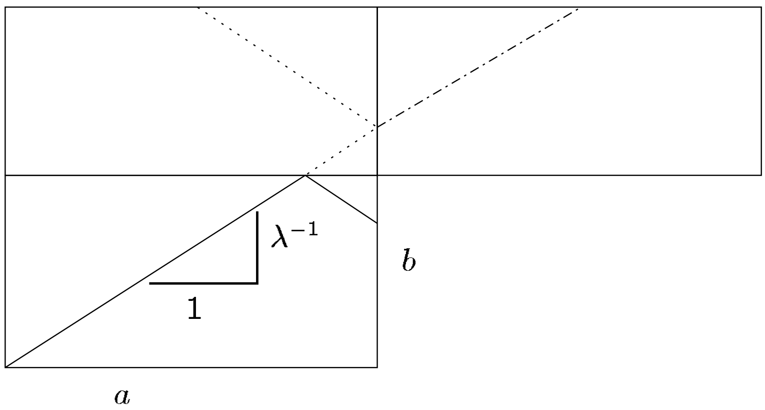


Poincaré, John, Schaeffer: billiard problem with relevant quantity *the rotation number* $\rho =$??

- Irrational $\rho \implies$ Characteristics close, smooth solutions possible.

- Rational $\rho \implies$ Characteristics towards attractors, or no solutions.

Example: The $a \times b$ rectangle



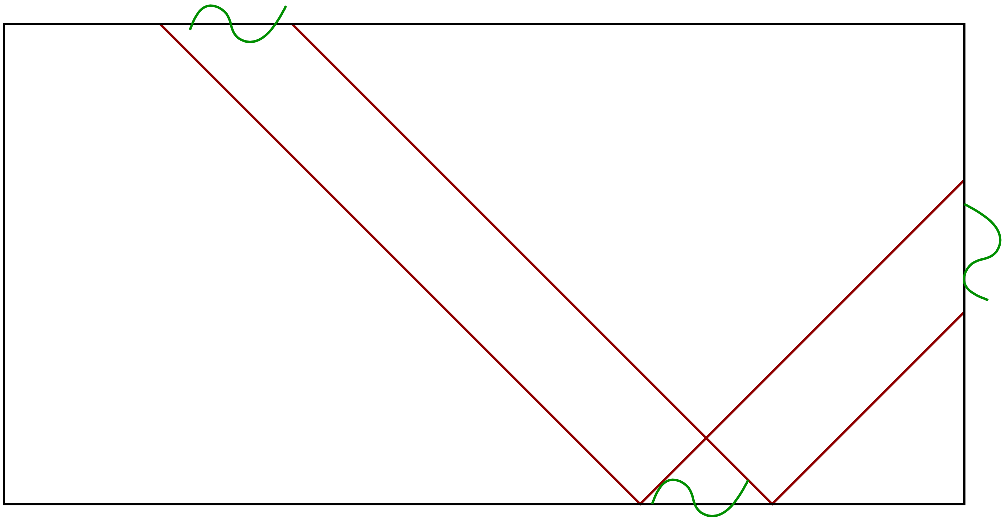
from the picture $j = na$ and $j\lambda^{-1} = mb$,

$$\lambda = \frac{mb}{na}.$$

General solution: separation of variables.

$$\Psi(x,z) = \sum_{j=0}^{\infty} a_n \sin(jm\pi x/a) \sin(jn\pi/b),$$

Shows ill-posedness and *fundamental interval*.



How to obtain uniqueness ?

- **Fill the fundamental interval.** Unknown, hard to discretise, no 3D analogue, not physical.

- **Minimise the energy.** Still not unique, set $||\Psi|| = 1$. Physical interpretation !

Example: The trapezoid

Has fractal structure, attractors, ill-posedness.
Exact solutions not known.

Limitpoint Diagram

Discretisation

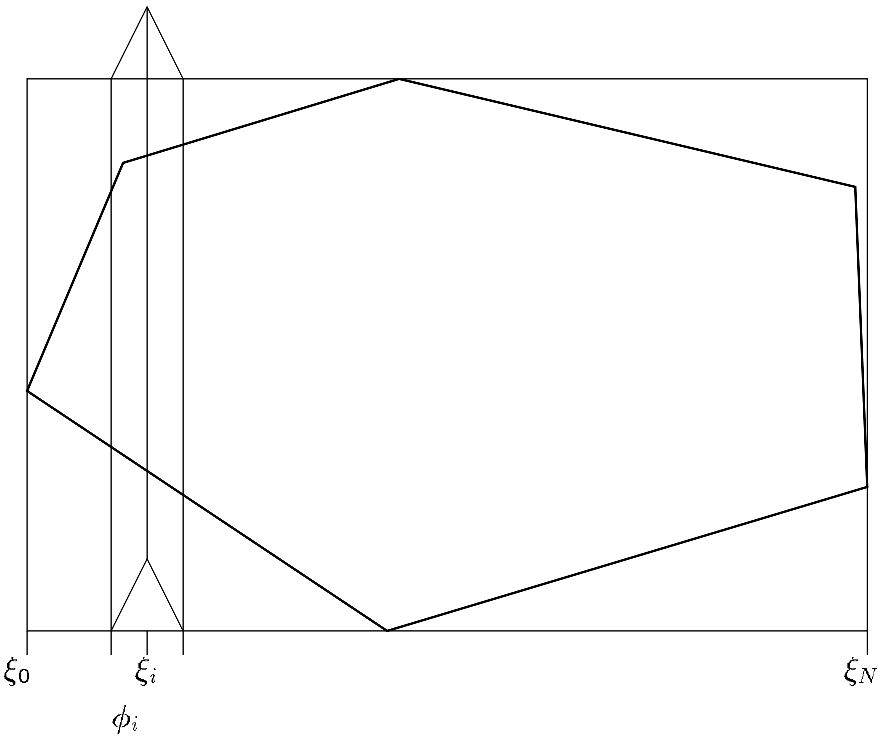
The equations

$$\Psi(\xi,\eta) = \mathcal{F}(\xi) + \mathcal{G}(\eta) \quad \text{in} \quad \bar{\Omega},$$
$$\Psi(\xi,\eta) = \mathcal{F}(\xi) + \mathcal{G}(\eta) = 0 \quad \text{at} \quad \partial\bar{\Omega}.$$

suggest taking

$$\mathcal{F}(\xi) = \sum_{i=0}^{N} f_i \psi_i(\xi) \ \in \ H^1,$$
$$\mathcal{G}(\eta) = \sum_{j=0}^{M} g_j \phi_j(\eta) \ \in \ H^1,$$

We take for ψ_i,ϕ_j the 'hatfunctions' :



- Is an **exact** solution to the Poincaré equation.

- Grid refinement is easy.

Boundary Element Method

Galerkin projection of $\Psi = 0$:

$$\int_{\partial\Omega} (\mathcal{F} + \mathcal{G}) v dl = 0$$

Take $v = \phi_i$, $v = \psi_j$ to get

$$\mathbf{A}\mathbf{x} = \mathbf{0},$$

with $\mathbf{x} = (f_0, \ldots, f_N, g_0, \ldots, g_M)$.

- Exact solution for a *perturbed boundary*.
- Piecewise linear boundaries fit potentially.

Minimal energy:

$$\begin{aligned} E^2 &= \int_{\bar{\Omega}} u^2 + v^2 d\xi d\eta \\ &= \frac{\lambda}{8} \int_{\bar{\Omega}} (\mathcal{G}_\eta + \mathcal{F}_\xi) + \lambda^2 (\mathcal{G}_\eta - \mathcal{F}_\xi) d\xi d\eta \end{aligned}$$

can be written

Minimise $||\mathbf{Lx}||_2^2$.

The normalisation $||\Psi|| = 1$ becomes $\int_{\bar{\Omega}} (\mathcal{F} + \mathcal{G})^2 d\xi d\eta = 1$ and can be discretised as

$$||\mathbf{Nx}|| = 1$$

If we integrate over enclosing rectangle of $\bar{\Omega}$, then:
All matrices are of size $\propto N + M$ and can be computed in $O(N + M)$ time.

Tikhonov Regularisation

Finding a compromise between minimising the energy and the residual:

$$x_\nu = \min_x \{||Ax||_2^2 + \nu ||Lx||_2^2\}. \qquad (1)$$

The L-curve criterion gives in some sense an 'optimal' solution [plaatje van L-curve].

Rewrite:

$$x_\nu = \min_x || \begin{pmatrix} A \\ \nu L \end{pmatrix} x||_2^2 \quad (2)$$

We want to normalise:

$$||Nx||_2^2 = 1. \quad (3)$$

Set $Nx = y$ and solve

$$y_\nu = \min_y || \begin{pmatrix} AN^{-1} \\ \nu LN^{-1} \end{pmatrix} y||_2^2 \quad (4)$$

Solve by SVD, which has solutions with $||y||_2^2 = 1$.

Results: The rectangle

Results: The trapezoid