Internal Gravity Waves in Stratified Fluids.

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Governing Equations I.

Internal waves *in* a fluid volume supported by

- Density Stratification ($\frac{d\rho(z)}{dz} \neq 0$)
- Rotation ($\mathbf{f \neq 0}$)

The momentum equation in a rotating frame:
Newton,Pressure gradient,Coriolis,gravity

$$
\begin{aligned}
\rho^*(\frac{\partial u}{\partial t} - fv) &= -\frac{\partial p}{\partial x} && (1)\\
\rho^*(\frac{\partial v}{\partial t} + fu) &= -\frac{\partial p}{\partial y} && (2)\\
\rho^*\frac{\partial w}{\partial t} &= -\frac{\partial p}{\partial z} - \rho' g && (3)
\end{aligned}
$$

Governing Equations II.

Plus mass conservation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\frac{\partial \rho(z)}{\partial t}-w\frac{\rho^*}{g}N^2(z) = 0 \tag{5}$$

with *inertial frequency*

$$N^2(z) = -\frac{g}{\rho^*}\{\frac{d\rho_0}{dz}+\frac{\rho_0 g}{c_s^2}\} > 0 \tag{6}$$

Look for time-periodic solutions

$$\phi \propto e^{i\omega t} \tag{7}$$

then if N is constant:

$$\boxed{p_{xx}+p_{yy}-(\tfrac{\omega^2-f^2}{N^2-\omega^2})p_{zz} = 0} \tag{8}$$

Governing Equations III.

At the boundary $\partial\Omega$, demand $\mathbf{u}\cdot\mathbf{n} = 0$, this gives

$$\boxed{\begin{array}{rcr} p_{xx} + p_{yy} - \lambda p_{zz} & = & 0 \quad \text{in } \Omega \\ (p_x, p_y, -\lambda p_z)\cdot\mathbf{n} & = & 0 \quad \text{on } \partial\Omega \end{array}} \tag{9}$$

Properties of the **Poincaré equation** or **Sobolev equation**

- Hyperbolic second order
- Linear
- Wave equation in spatial coordinates

Method of Characteristics.

Two families of characteristic lines:

$$\xi(x,z) = x - \sqrt{\lambda}^{-1} z = c_1 \tag{10}$$

$$\nu(x,z) = x + \sqrt{\lambda}^{-1} z = c_1 \tag{11}$$

$$\tag{12}$$

In these coordinates:

$$\boxed{\begin{array}{c} p_{\xi\nu} = 0 \\ p(\xi,\nu) = \mathcal{F}(\xi) + \mathcal{G}(\nu) \end{array}} \tag{13}$$

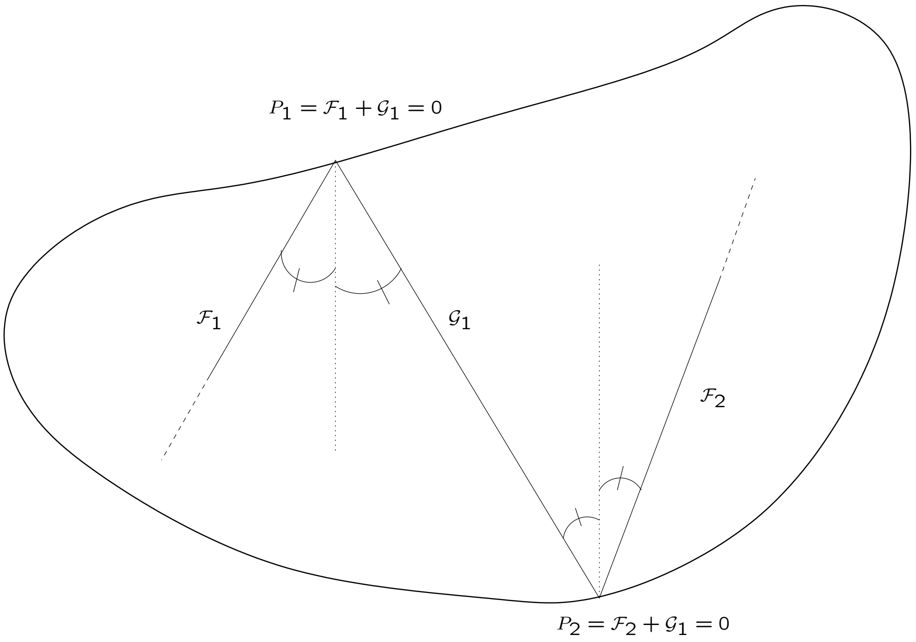
On a **characteristic convex** domain Ω.
Energy propagates along characteristics,
what happens at the boundary ?

At the boundary.

Use the streamfunction: $\Psi_z = u,\ \Psi_x = v$

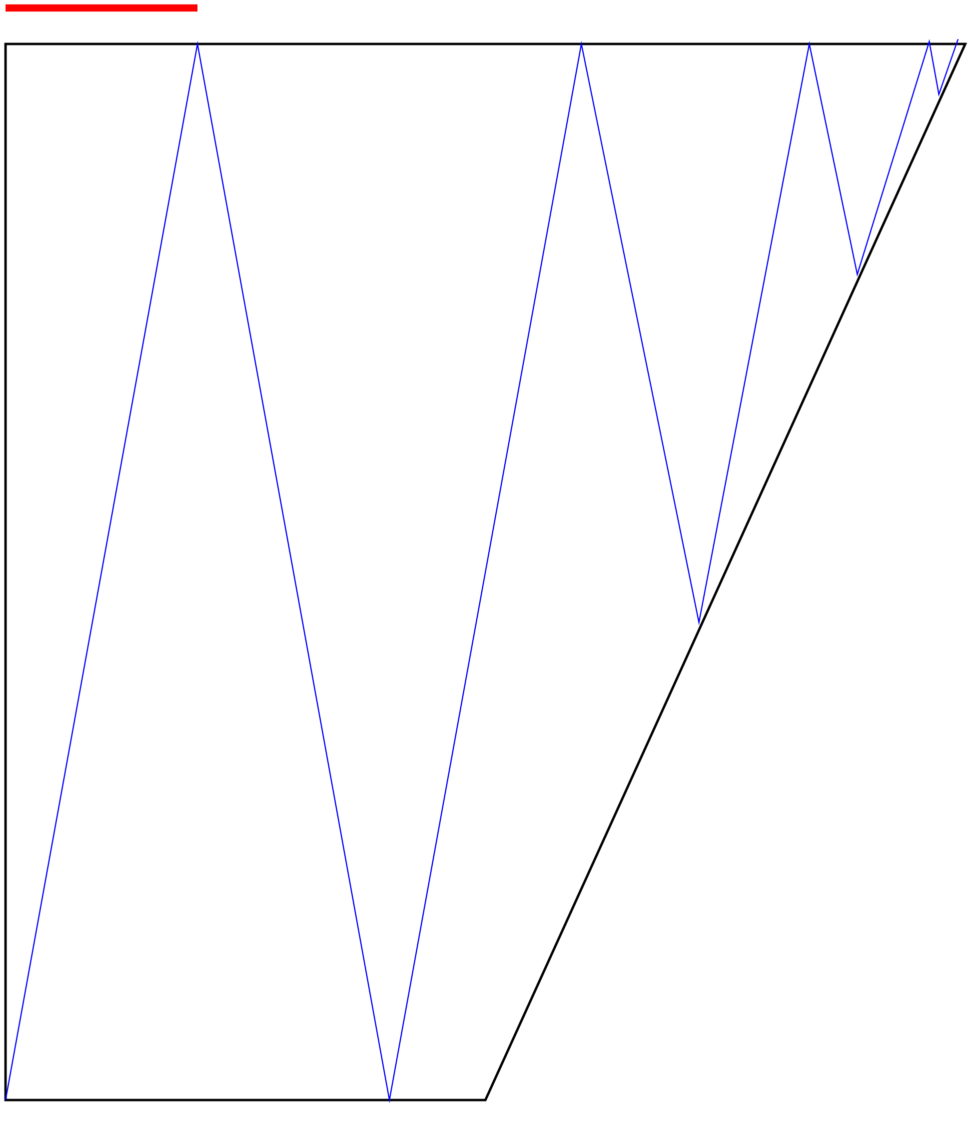
$$\boxed{\begin{array}{rcll} \Psi(\xi,\nu) & = & \bar{\mathcal{F}}(\xi)+\bar{\mathcal{G}}(\nu) & \text{in } \Omega \\ \Psi(\xi,\nu) & = & 0 & \text{on } \partial\Omega \end{array}} \qquad (14)$$

Leads to invariance of partial pressure.



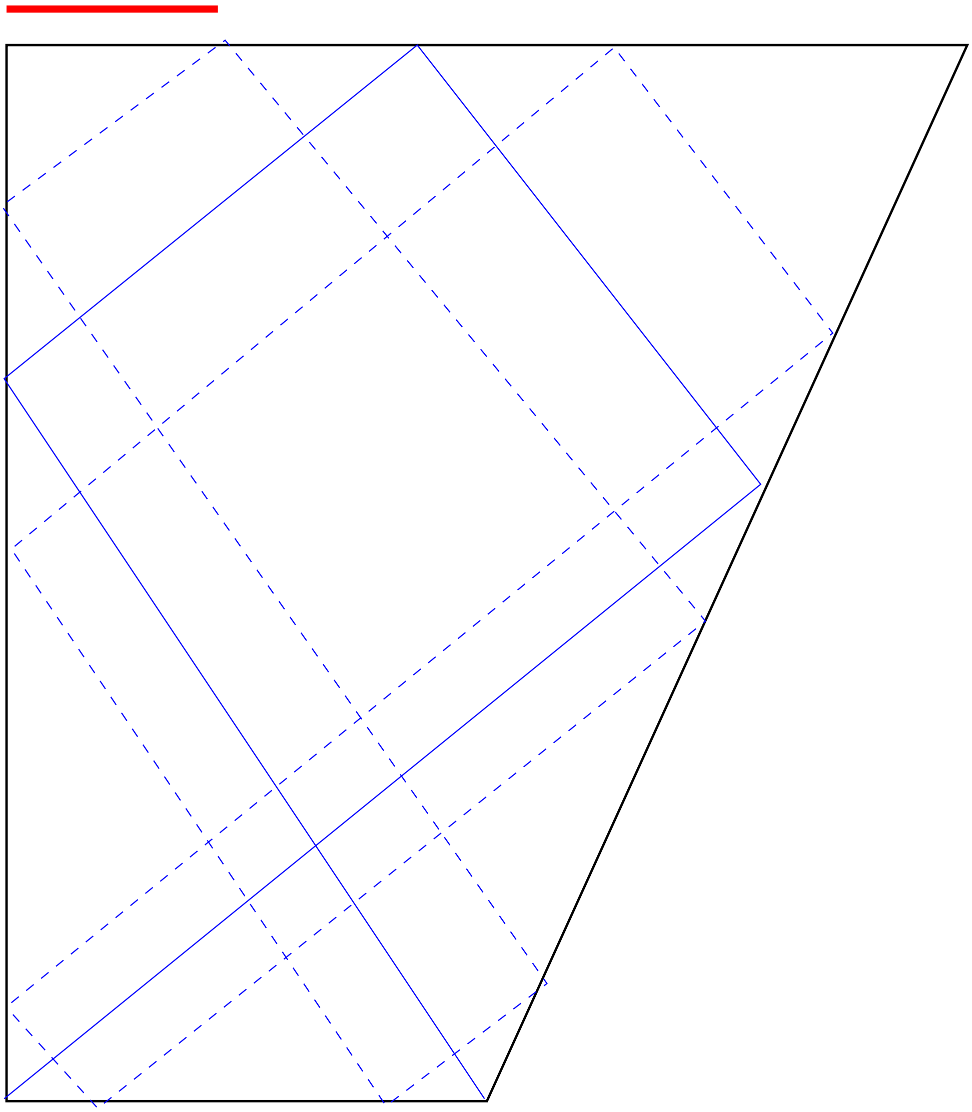
Model Geometry; the Trapezoid.

A **point attractor**.



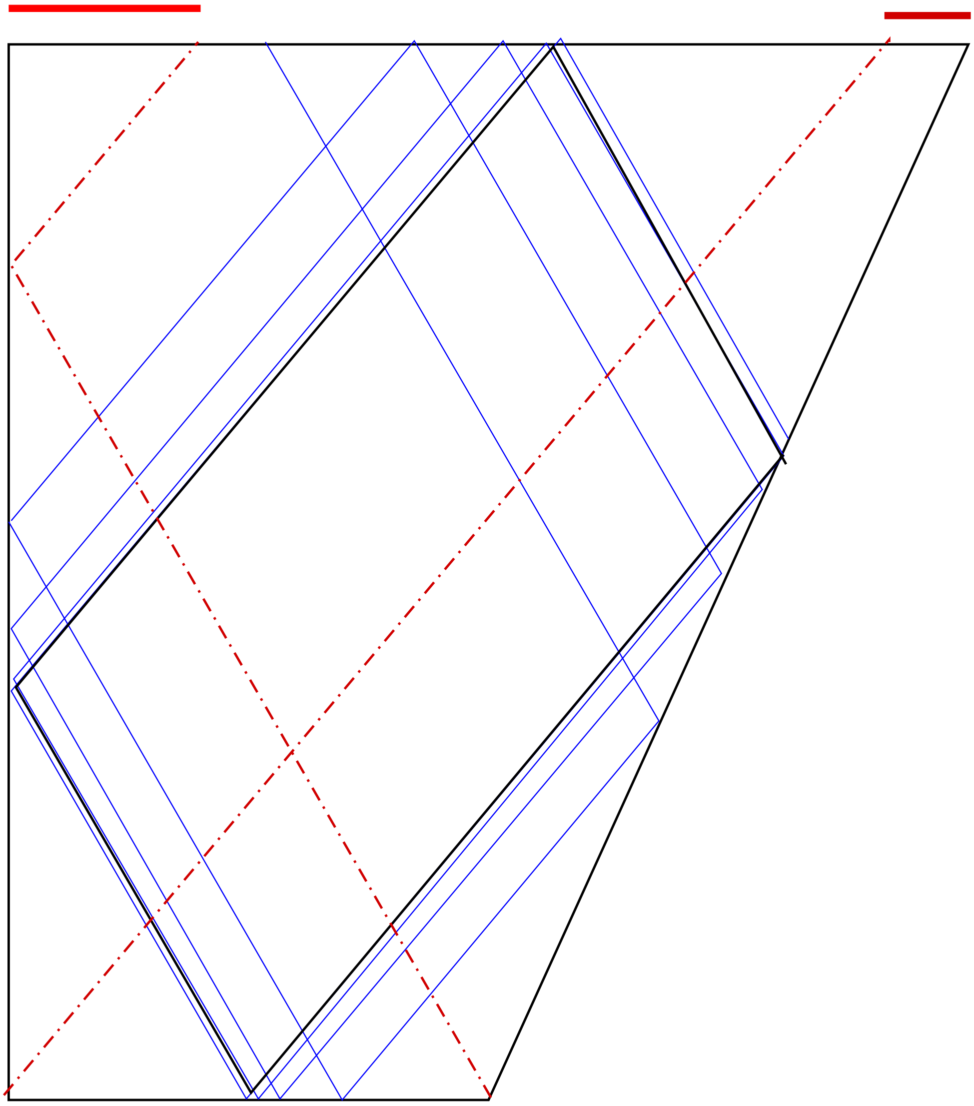
Model Geometry; the Trapezoid.

A global resonance (modal solution, seiche).

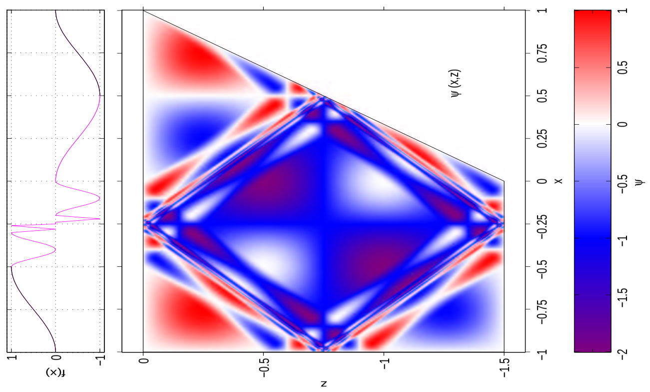


Model Geometry; the Trapezoid.

An (1,1)-**attractor**.



Fractal Structure.



Ordering of Solutions in the spectrum.

'Ill-posedness'

Physical Expirimental Data.

Numerical Problems.

- Continuity of the spectrum.
  - Global resonances at points.
  - Attractors in intervals.
- Infinite multiplicity → apply Dirichlet b.c.
- Fractal structure.

Try **Finite Elements** ...

Standard Finite Elements.

- Find $p \in H^1$ such that

  $$\boxed{a(p,v) = \int_\Omega p_x v_x - \lambda p_z v_z dO = 0, \forall v \in H^1}$$

- Triangulate your domain Ω.

- Choose the basis
  $\phi_j(N_i) = \delta_{i,j}$,

- Express p in terms of this basis:
  $p_h(x,y) = \sum_{j=1}^m p_h(N_j)\phi_j(x,y)$

Then:

$$\begin{aligned} a(p,v) &\approx a(p_h,\phi_i) \\ &= \sum_{j=1}^{m} p_h(N_j)a(\phi_j,\phi_i), i = 1,\ldots,m \end{aligned}$$

The Discretised System.

The continuous equation

$$\boxed{\begin{array}{rcll} p_{xx} - \lambda p_{zz} & = & 0 & \text{in } \Omega \\ (p_x, -\lambda p_z) \cdot \mathbf{n} & = & 0 & \text{on } \partial\Omega \end{array}} \tag{15}$$

then yields:

$$\boxed{(\mathbf{A} + \lambda \mathbf{B})x = 0} \tag{16}$$

- Boundary conditions / Fundamental interval ?
- Infinite kernell lost, what solutions are found ?
- λ Eigenvalue or parameter ?
- No error estimate.

A Finite Element Solution.

An $(1,1)$-attractor.

A Finite Element Solution.

A global resonance.

Sugestions.

- Regularisation (using SVD).
- Note: usual multistep schemes not applicable.
- Other discretisations (BEM ?).
- Dynamical systems approach ?
- Add viscosity $\longrightarrow$ elliptic problem.

Problem reduction.

Transform to characteristic coordinates.
At boundaries:

$$
\begin{aligned}
(p_x, -p_y) \cdot (n_x, n_y) &= 0 \qquad (17)\\
(f'(\xi), g'(\nu)) \cdot (n_\xi, n_\nu) &= 0 \qquad (18)
\end{aligned}
$$

Normal at boundaries $h_{1,2}(\xi)$ given by

$$\mathbf{n} = (-h'_{1,2}(\xi), 1)$$

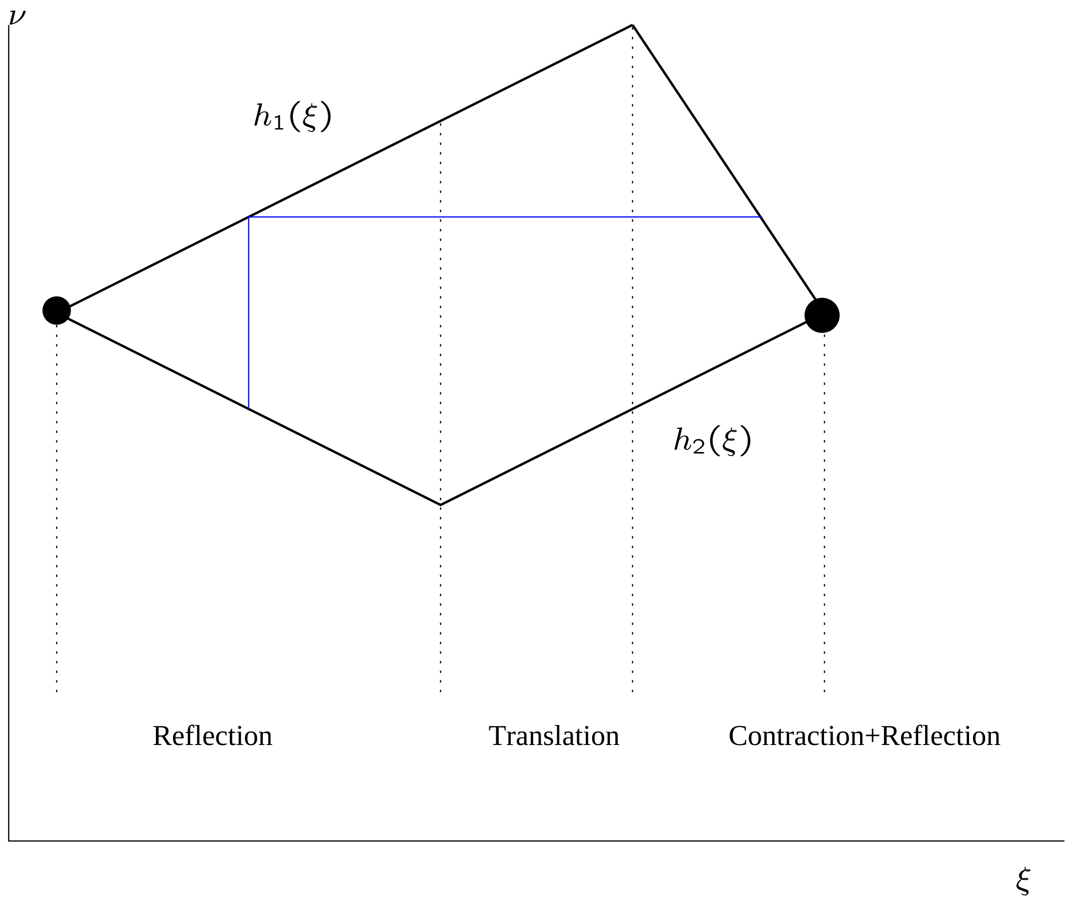
This gives at the boundary $\nu = h_{1,2}(\xi)$

$$-f'(\xi)h'_{1,2} + g'(h_{1,2}) = 0$$

Chainrule:

$$g(h_1)(\xi) = g(h_2)(\xi)$$

The Boundary as a Dynamical System.



- Gives a mapping $T : \xi \longrightarrow \xi$.

- Discretise by approximating $g(\xi)$?