

A finite element method for internal gravity waves

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Overview

- **Physics**

Leads to the eigenvalue problem

$$p_{xx} + \lambda p_{zz} = 0,$$
$$(p_x, \lambda p_z) \cdot \mathbf{n} = 0.$$

- **Method of characteristics**

Indicates the structure of the solution.

- **The rectangle & the trapezoid**

The 2D geometries studied.

- **The finite element method**

Discretisation of the problem \rightarrow
 $(\mathbf{A} + \lambda \mathbf{B})\mathbf{x} = 0.$

- **The Warp program**

Demonstration of the software.

Surface waves:

density transition water-air.

Internal gravity waves:

density gradients *in* the water volume.

Approximations:

linearisation, quasi incompressibility and *Boussinesq* lead to:

$$\rho^* \left(\frac{\partial u}{\partial t} - fv \right) = - \frac{\partial p'}{\partial x}$$

$$\rho^* \left(\frac{\partial v}{\partial t} + fu \right) = - \frac{\partial p'}{\partial y}$$

$$\rho^* \frac{\partial w}{\partial t} = - \frac{\partial p'}{\partial z} - \rho' g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} - w \frac{\rho^*}{g} N^2 = 0$$

The factor N is given by

$$N^2(z) \equiv - \frac{g}{\rho^*} \left\{ \frac{d\rho_0}{dz} + \frac{\rho_0 g}{c_s^2} \right\} > 0$$

Assumptions:

- $\phi \propto e^{i\omega t}$
- ρ_0 such that $N(z)$ is a constant.
- No rotation: $f = 0$.
- 2 dimensional bounded domain Ω .
- No-flow boundary conditions: $\mathbf{u} \cdot \mathbf{n} = 0$.

After some calculation:

$$\frac{\partial^2 p}{\partial x^2} + \lambda \frac{\partial^2 p}{\partial z^2} = 0 \quad \text{on } \Omega$$
$$\mathbf{n} \cdot \left(\frac{\partial p}{\partial x}, \lambda \frac{\partial p}{\partial z} \right) = 0 \quad \text{on } \partial\Omega$$

With:

$$\lambda \equiv \frac{\omega^2}{\omega^2 - N^2}, \quad 0 \leq \omega^2 \leq N^2$$

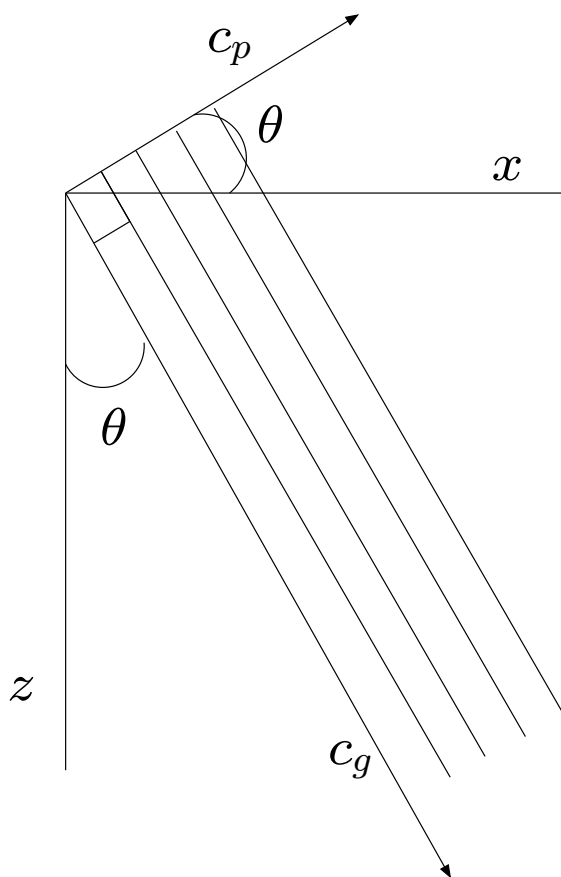
Hyperbolic second order partial differential equation.

The dispersion relation

Relation between the frequency ω and wavevector $\mathbf{k} = (k, m)$

$$\omega^2 = N^2 \cos(\theta)$$

The angle is *fixed* for a given frequency !



Method of characteristics

Two families of characteristic lines:

$$\xi(x, z) = x + (-\lambda)^{-1/2}z = c_1$$

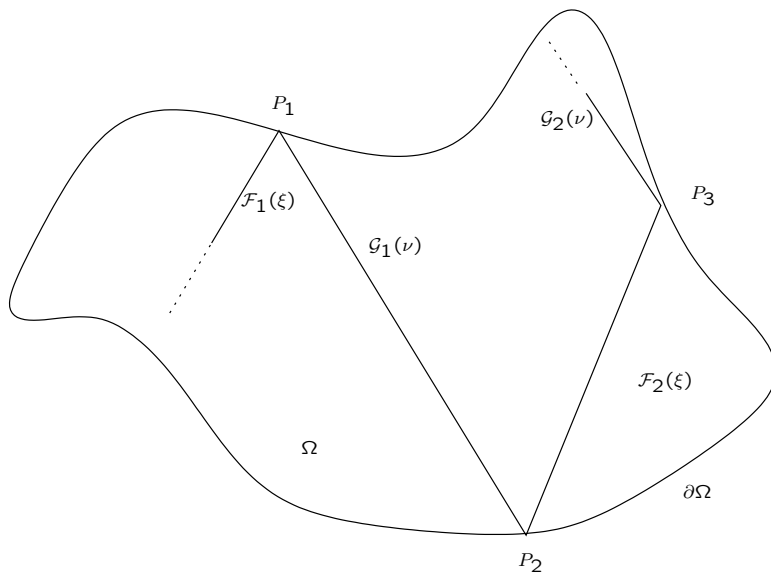
$$\nu(x, z) = x - (-\lambda)^{-1/2}z = c_2$$

$$p(x, z) = \mathcal{F}(\xi) + \mathcal{G}(\nu)$$

Angle with vertical: $\theta = \arccos(\omega^2/N^2)$!

→ Energy propagation along characteristics.

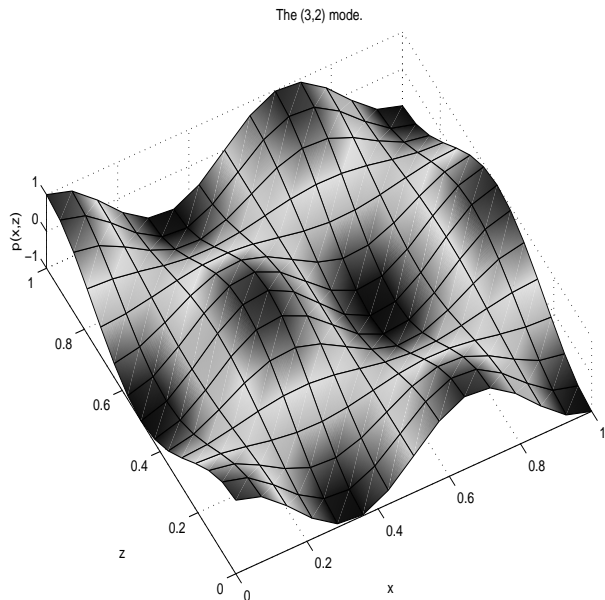
→ Two characteristics joining on $\partial\Omega$:
reflection.



Conclude: $\mathcal{F}_1 = \mathcal{G}_1 = \mathcal{F}_2 = \mathcal{G}_2 = \dots$

Transport of pressure on the boundary.
Boundary determines inner domain.

The rectangle



$$p(x, z) = A \cos\left(\frac{k\pi x}{a}\right) \cos\left(\frac{l\pi z}{b}\right)$$

Eigenvalues given by:

$$\lambda = -\left(\frac{kb}{la}\right)^2$$

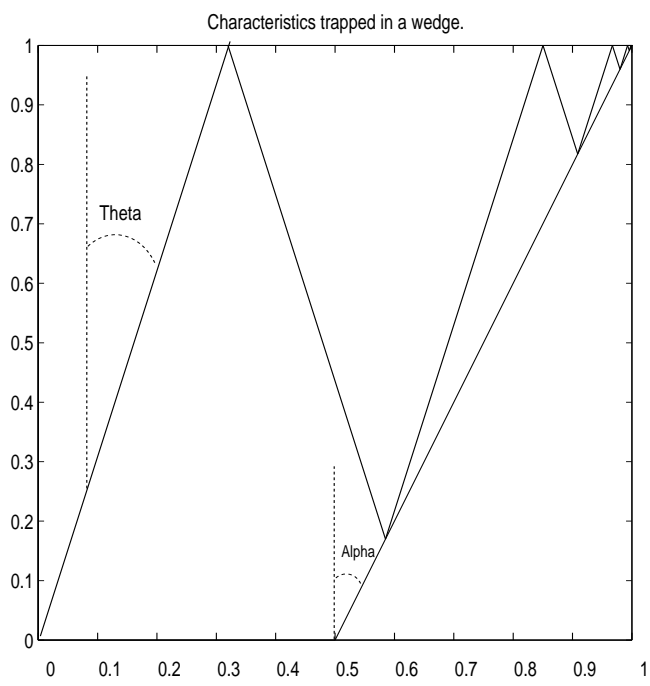
Solutions on $M \times N$ grid, stepsizes h_1, h_2 :

$$p(x_i, z_j) = A \cos\left(\frac{k\pi}{h_1 N}\right) \cos\left(\frac{l\pi}{h_2 M}\right)$$

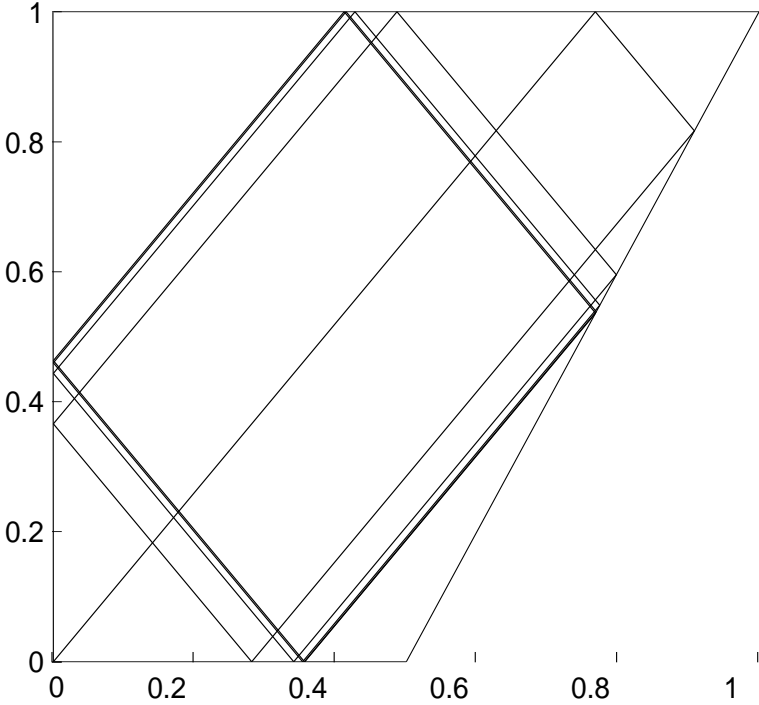
With eigenvalues:

$$\tilde{\lambda} \approx \left(1 + \frac{\pi^2}{8} \left(\frac{l^2}{N^2} - \frac{k^2}{M^2}\right)\right) \lambda$$

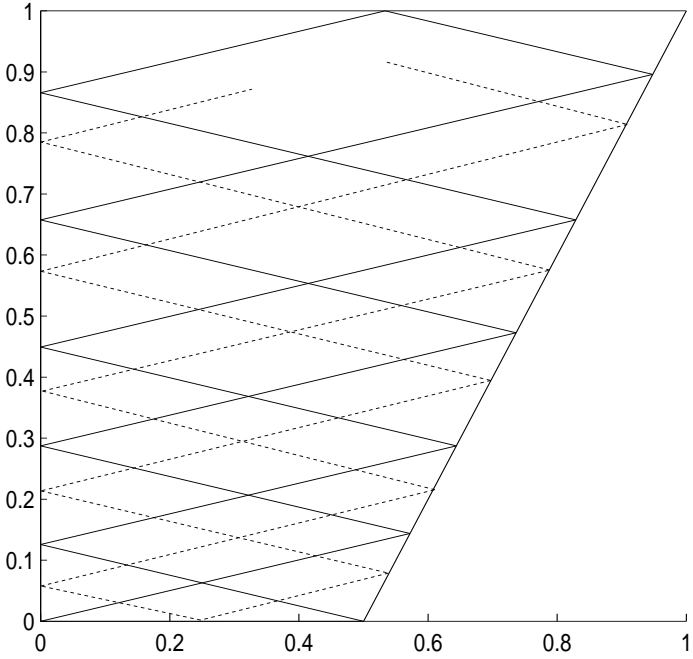
The trapezoid



Characteristics focussing onto an attractor.



Characteristics folding back upon themselves, a global resonance.



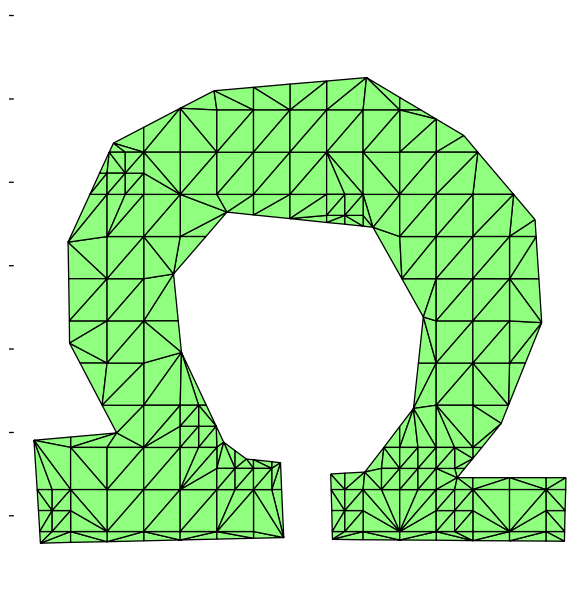
Finite elements

- **Step I**, variational formulation

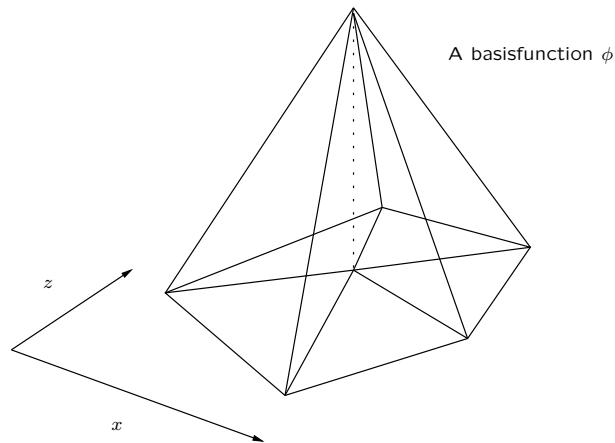
$$a(p, v) = \int_{\Omega} p_x v_x + \frac{\omega^2}{\omega^2 - N^2} p_z v_z dO = 0$$

This incorporates the Neumann boundary conditions.

- **Step II**, triangularisation
Using the QMG 2.0 mesh generator.



Step III, Construct the finite element equations



- Take for the testfunctions v the basisfunctions:

$$\phi_j(N_i) = \delta_{ij} \text{ for a node } N_j.$$

- Approximate p by a series:

$$p_h(x, y) = \sum_{j=1}^m p_h(N_j) \phi_j(x, y)$$

Then:

$$a(p, v) = a(p, \phi_i) \approx a(p_h(x, y), \phi_i) = \sum_{j=1}^m p_h(N_j) a(\phi_j, \phi_i) = 0, \quad i = 1, \dots, m$$

The generalised eigenproblem

In matrix-vector form:

$$\mathbf{M}\mathbf{x} = \mathbf{0}$$

with

$$m_{ij} = a(\phi_i, \phi_j), \quad x_i = p_h(N_i)$$

Split:

$$\mathbf{M}\mathbf{x} = \mathbf{A}\mathbf{x} + \frac{\omega^2}{\omega^2 - N^2}\mathbf{B}\mathbf{x} \equiv (\mathbf{A} + \lambda\mathbf{B})\mathbf{x}$$

Generalised eigenproblem:

$$(\mathbf{A} + \lambda\mathbf{B})\mathbf{x} = \mathbf{0}$$

Non-trivial solution

→ $\det(\mathbf{A} + \lambda\mathbf{B}) = 0$

→ Linear dependence

→ Underspecified system

Satisfy boundary conditions by combining eigenvectors, adding restraints. Correspondence to fundamental interval ?